PHYSICAL REVIEW E VOLUME 61, NUMBER 5 MAY 2000

Hysteresis at low Reynolds number: Onset of two-dimensional vortex shedding

V. K. Horváth, J. R. Cressman, W. I. Goldburg, and X. L. Wu
Department of Physics and Astronomy, University of Pittsburgh, Pittsburgh, Pennsylvania 15260

(Received 7 October 1999)

Hysteresis has been observed in a study of the transition between laminar flow and vortex shedding in a quasi-two-dimensional system. The system is a vertical, rapidly flowing soap film which is penetrated by a rod oriented perpendicular to the film plane. Our experiments show that the transition from laminar flow to a periodic von Kármán vortex street can be hysteretic, i.e., vortices can survive at velocities lower than the velocity needed to generate them.

PACS number(s): 47.27.Gs, 67.40.Vs, 68.15.+e, 92.60.Ek

Pattern forming fluid flow is often difficult to understand even at quite modest velocities. A widely studied example is the chain of vortices generated downstream from a rod placed in a uniform flow, with the axis of the rod oriented perpendicular to the flow direction. In a pioneering study of this problem, von Kármán analyzed the stability of this vortex street [1]. His starting point was the assumption that the flow is two-dimensional, i.e., no velocity variations are generated along the rod axis. He was concerned only with the stability of the vortex array and made no attempt to explore the transition itself. Although researchers have put considerable effort to better understand this phenomenon, which is considered to be a precursor to turbulence, a full understanding is still lacking [2].

It is now widely accepted that the transition (primary instability) from laminar flow to vortex shedding is supercritical in nature; on changing the mean flow speed $\overline{V}$ around the transition point, vortex shedding appears or disappears at a unique velocity V_c. Below (above) V_c the only stable state of the system is that of the laminar (vortex shedding) flow.

In an important wind tunnel experiment on vortex shedding, Provansal *et al.* [3] showed that the primary instability occurs at a critical Reynolds number $\mathrm{Re}_c \simeq 46$. The vortex shedding disappeared at the same velocity at which the onset occurred, i.e., hysteresis was absent. These measurements are well described by the Landau-Stuart theory for supercritical transitions. Sreenivasan *et al.* [4] observed that an acoustic disturbance or a vibration of the rod itself could activate the vortex street when Re was less than Re_c but greater than some minimum value $\mathrm{Re}_{min} \simeq \mathrm{Re}_c/2$. Their experiment showed that the general behavior of the system was position independent. The Landau-Stuart theory was adequate to explain their experimental findings, too. Couder and Basdevant [5] and Gharib and Derango [6] were among the first to study vortex shedding behind a cylinder in soap films. In particular, Gharib and his associates carried out systematic investigations of the vortex shedding frequency f as a function of $\overline{V}$ in a horizontally flowing film [6].

In this Rapid Communication we present measurements demonstrating that the primary instability can be hysteretic. This finding is also in contrast with numerical results [7]. Although hysteretic transitions are well known in fluid dynamics [8], we know of no published laboratory experiments or computer simulations, which suggest that the onset of vortex shedding from a rigid rod is a hysteretic phenomenon.

Our experimental setup consists of a rapidly flowing soap film formed between two vertically positioned 0.25 mm diameter nylon lines separated by 6 cm (see Fig. 1). The lines are parallel along a 45 cm vertical segment. They are joined at both ends of the segment. The film is fed continuously from the top through a high precision metering valve. The valve was opened and closed at different rates by a motor connected to it, thus the acceleration $d\overline{V}/dt$ was under experimental control. The film is a mixture of soap (Dawn) and distilled water with a typical soap concentration of 1 wt %.

Using an infrared absorption technique, the soap film thickness profiles parallel and perpendicular to $\overline{V}$ were measured. In the absence of the rod, the horizontal film thickness was essentially flat with less than 5% variations, whereas there was systematic thinning along the vertical direction. Due to a small acceleration of the film, the latter amounts to a 1% change in the thickness over a distance between the two LDV probes. The average thickness, which was found to be several microns, does not change over the range of $\overline{V}$ spanned by our experiments.

The velocity measurements were made using a dual head laser Doppler velocimeter (LDV). At the center of the parallel segment of the soap film, a glass rod of diameter $d = 1$ mm penetrates the film in the z direction. The velocities $\overline{V}$ and the fluctuating horizontal component $V_x(t)$ were mea-

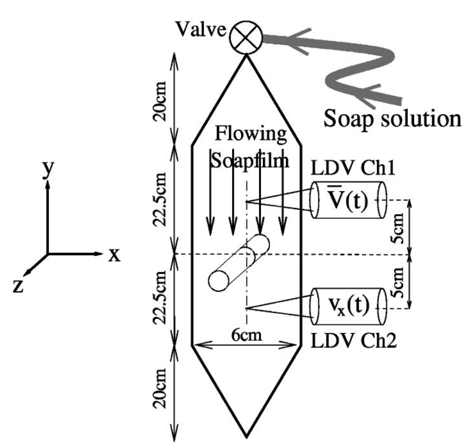

FIG. 1. Schematic of the experimental setup. The diameter of the rod is 1 mm.

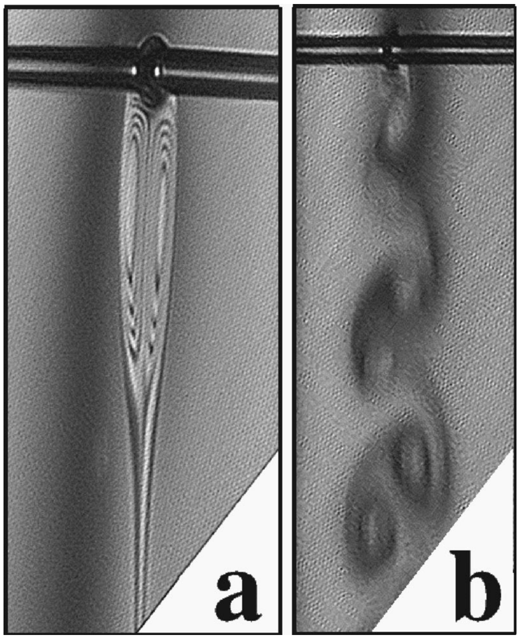

FIG. 2. Interference pictures of the flowing soap film below the rod in (a) the laminar state and (b) the vortex shedding state. It was shown previously that the thickness of the film is correlated with the film velocity [10].

sured at points 5 cm above and 5 cm below the rod, respectively.

At a certain critical flow rate there is a transition from the laminar flow (LF) state to the vortex shedding (VS) state. Slightly below this critical velocity, the flow is characterized by two counter-rotating vortices underneath the rod [see Fig. 2(a)]. In the VS state these recirculating vortices peel off the rod and flow downstream. The continuously generated counter-rotating vortices form a vortex street [see Fig. 2(b)], as described by von Kármán [1].

All measurements were made near the transition. The experiments start with a value of $\overline{V}$ corresponding to the LF regime. The valve is then opened to slowly increase $\overline{V}$. The critical velocity where VS commences is called V_c^{up}. After some waiting time the valve is slowly closed. As a result $\overline{V}(t)$ decreases, and at $\overline{V}(t)=V_c^{down}$ the system undergoes a transition from VS into LF. This cycle was repeated several times in each run. To find the transition from LF to VS, $V_x(t)$ was recorded together with $\overline{V}$. In theory $V_x(t)$ is constant in the laminar state and equal to zero directly below the center of the rod. In the VS regime $V_x(t)$ is expected to display periodic behavior.

In order to determine the transition velocity accurately we have calculated the velocity probability distribution function $P(V_x)$. For this purpose the total recording time was segmented into intervals $\Delta t \simeq 200$ ms. Then $P(V_x)$ was calculated within each interval. The intervals Δt were much smaller than the characteristic changing time of $\overline{V}(t)$ but large enough to include a statistically significant number of data points and many periods of oscillation in the VS state. A typical result is shown in Fig. 3. Here the magnitude of $P(V_x,t)$ is mapped into gray scale values and shown as a function of V_x and t. It can be seen that sharp changes in $P(V_x,t)$ precisely indicate the transition between the LF and VS regimes. In the LF regime, P is sharply peaked at $V_x=0$, but in the VS regime P has a broad distribution, as for a (noisy) sine wave. These measurements provide a powerful way to accurately determine the times t_d^* and t_u^* of LF $\rightarrow$VS and VS$\rightarrow$LF transitions [9].

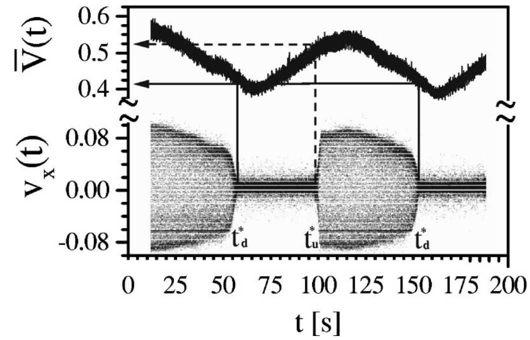

FIG. 3. Here the function $\overline{V}(t)$ is shown together with the $P(v_x,t)$ map at $|d\overline{V}/dt|=0.004$ m/s^2. The gray scale corresponds to the probability density of v_x. The solid and dashed lines are intended to guide the eye for the transitions VS$\rightarrow$LF and LF$\rightarrow$VS.

The hysteresis is apparent in Fig. 3, where $\overline{V}(t)$ and $P(V_x,t)$ are displayed together. By determining the value of $\overline{V}(t)$ at the transition times t_u^* and t_d^* one finds that $V_c^{up}=\overline{V}(t_u^*)$ is not equal to $V_c^{down}=\overline{V}(t_d^*)$; that is, there is no unique critical velocity for the transition. In this analysis we have incorporated the fact that $\overline{V}$ is noisy at the level of $\pm 4\%$. Therefore, to determine V_c^{down} (and V_c^{up}) we always used the lowest (and highest) values of (the fluctuating) $\overline{V}$ at the transition.

The critical velocities were measured in 20 experimental runs. The ensemble average of the different experimental runs provide $\langle V_c^{down}\rangle = 0.42 \pm 0.02$ m/s. The relative difference $(\langle V_c^{up}\rangle - \langle V_c^{down}\rangle)/\langle V_c^{down}\rangle$ is approximately 14%. The Reynolds number—which is defined as $\mathrm{Re}=Vd/\nu$—is roughly 50 in the hysteretic gap. A large uncertainty of this value comes from the poorly determined two-dimensional soap film viscosity ν, which depends on the film thickness [10].

Next, we demonstrate that the observed hysteresis is *not* a result of some delay in the response of the system to the bifurcation, as it would be the case for dynamical hysteresis [11,12]. We set the mean flow rate constant in the interval $[V_c^{down}, V_c^{up}]$. In this case the laminar state can persist for an indefinite length of time. However, applying sufficiently large acoustic or pulselike mechanical perturbations, the system could be driven into the VS state. The system remains in this state even if the perturbation is turned off. Applying similar perturbations, the VS$\rightarrow$LF transition was never observed at constant $\overline{V}$. As a result we can conclude that we observed static hysteresis, i.e., its existence is independent of the rate of change of $\overline{V}$.

At the transition, a certain time interval is required for the new steady state to establish itself as it is seen in the close vicinity of t_u^* in Fig. 3. The overall time of the transient remains finite even if $\overline{V}$ is changed very slowly. This would not be true for continuous transitions [13]. Careful examina-

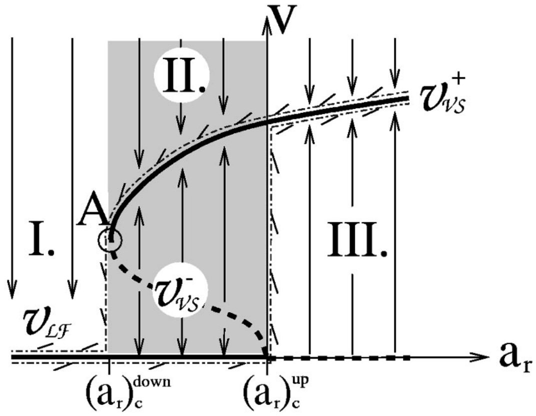

FIG. 4. Bifurcation diagram of Eq. (2). The turning point of the nontrivial solution is indicated by *A*. The solid and dashed lines designate the stable and unstable solutions respectively.

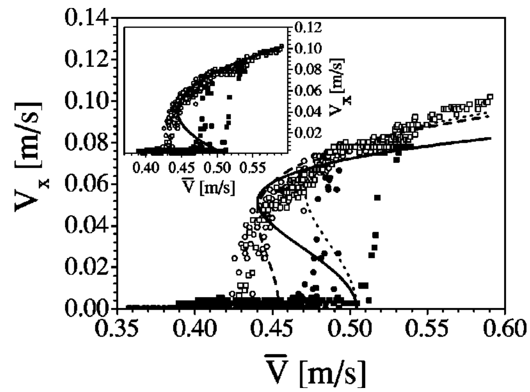

FIG. 5. Main figure shows our experimental data together with three different fits using Eq. (3). The circles and squares represent two different data sets. Closed and opened symbols are used for data taken at increasing and decreasing $\overline{V}$ respectively. The solid and dashed lines have the same meaning as in Fig. 4. The inset shows our best parametric fit to the data (see text).

tion of $P(V_x, t)$ shows that the most probable V_x grows exponentially at the transition points. Using exponential curve fitting one finds that $\tau^{growth} \simeq 1.32 \pm 0.05$ s. Thus it was necessary to vary $\overline{V}$ slowly enough, that this finite transit time did not interfere with our identification of V_c^{up} and V_c^{down}.

Because hysteresis is not seen at the onset of vortex shedding in 3D systems, we have looked for possible side effects that might be responsible for our unexpected finding. It is known that oscillations of the rod can induce hysteresis [8]. Our rod is rigidly supported, so its vibration can only be induced by the fluid itself [14]. Since the film is only a few microns thick, it carries insufficient momentum flux to deflect our 1 mm diameter glass rod more than a nanometer. The possible motion of the vertical fishing lines which support the film was also considered. Taking into account the geometry of our experimental setup, we estimate that their deflection, produced by the passage of an eddy, is no more than a micron. Auxiliary experiments were also performed to exclude wetting properties and the effect of the air boundary near the film as possible sources of hysteresis [15].

Our experiments establish that the transition to vortex shedding can be subcritical. In the absence of any available theory, we consider a generic model based on an amplitude equation [16,17], namely, a fifth order Landau equation [13] for the complex velocity $\tilde{v} = v_x + i v_y$

$$d\tilde{v}(t)/dt = \tilde{a}\tilde{v} + 2\tilde{b}v^2\tilde{v} - \tilde{c}v^4\tilde{v}, \tag{1}$$

where all variables indicated with tilde are complex and $v_i = V_i - \langle V_i \rangle_t$ ($i = x$ or y). The complex velocity can be written as $\tilde{v} = v e^{i\phi}$, where ϕ is the phase factor and v is the amplitude. In this case the evolution of v is governed by the real part of Eq. (1),

$$dv/dt = v(a_r + 2b_r v^2 - c_r v^4), \tag{2}$$

where the subscript r is used to designate the real part of complex numbers and a_r is considered to be proportional to $(\overline{V} - V_c^{up})$. In the experiments a_r is adjusted by varying $\overline{V}$ [18].

In the following we consider the stationary solutions of Eq. (2). The laminar solution is $v_{LF} = 0$. Note that at the position where V_x is measured, $\langle V_x \rangle_t = 0$. Therefore, $v_x(t) = V_x(t)$ for that particular point. The time-dependent solution is described by

$$v_{VS}^{\pm\,2} = \frac{b_r}{c_r}\left(1 \pm \sqrt{1 + a_r c_r / b_r^2}\right). \tag{3}$$

It suffices to retain only the positive values of Eq. (3). Since the minus sign in this equation corresponds to an unstable state, the observable vortex shedding is represented by v_{VS}^{+}.

After some elementary algebra one can draw the stability (or bifurcation) diagram of this model (see Fig. 4, where only the positive solutions are shown). The solution v_{LF} is stable only if a_r is smaller than a critical value $(a_r)_c^{up} = 0$ (see regimes I and II). The solutions $v_{VS}^{\pm}$ exist only for $a_r \geqslant (a_r)_c^{down}$, where $(a_r)_c^{down} = -b_r^2/c_r$.

If $a_r < (a_r)_c^{down}$, then the system is in the LF state. On increasing a_r, the system enters into the hysteretic regime (II), where the system can stay in the LF state as long as the amplitude of the fluctuations does not exceed the value of v_{VS}^{-}. If $a_r > (a_r)_c^{up}$ then the LF state becomes unstable; even infinitesimally small noise will force the system to switch from v_{LF} to v_{VS}^{+}. In regime II, finite perturbations with amplitude larger than v_{VS}^{-} force the system to switch to the VS state. Such a switch is clearly inconsistent with a supercritical bifurcation, for which the saturation amplitude continuously changes with the control parameter. The everywhere stable solution v_{VS}^{+} does not exist below $(a_r)_c^{down}$. Therefore, the system must return from the VS state to the LF state at $a_r = (a_r)_c^{down}$ even when noise is absent. In this sense the VS→LF transition is more robust than its inverse.

Our observations are consistent with all the above described features of the model. However, the application of this model for data fitting purposes is not obvious, since the relationship between the parameters a_r, b_r, c_r and the control parameter of the experiment is unknown.

We use the above model and choose a_r to be $a_r = a_1 \epsilon$, where $\epsilon = (\overline{V} - V_c^{up})/V_c^{up}$. Other parameters are kept as fitting constants. In this case the relevant fitting parameters are

a_1/c_r, b_r/c_r, and V_c^{up}. Figure 5 shows the results of our fits together with the same experimental data set presented in Fig. 3. Shown here is the $\overline{V}$-dependence of the values of v_x for which $P(v_x)$ is maximum. This figure demonstrates both the sharpness of the VS→LF transition and the breadth of the LF→VS transition in our experiment. Open symbols show the measurements with decreasing $\overline{V}$ and closed symbols refer to measurements made in the opposite direction.

The dotted line in Fig. 5 is the best fit to the data with V_c fixed at a value of 0.504 m/s, which is the largest mean velocity where the laminar state was still observed. This fit seriously underestimates the width of the observed hysteretic gap. The dashed line is the best fit with a different constraint: it was required that the fit must terminate exactly at the lowest velocity for which the VS state exists. In this case the part of the curve which represents the unstable trajectory, terminates on the laminar branch in the middle of the hysteresis loop, predicting that LF state should not exist above $\overline{V}=0.45$ m/s. The solid line designates a fit where both of the above constraints are imposed.

Although the generic model of the hysteresis is in fair agreement with our observations, none of the above fits are very satisfactory. It is interesting to note that a simple three-parameter phenomenological equation $P_0(\overline{V}-V_c)+2P_1v-P_2v^2=0$ provides a surprisingly good fit to our data in the VS state. Using least square fitting to the solution of this equation provides the following result: $v_x=0.04\pm\sqrt{\overline{V}/39.06-0.0113}$, where the velocities are in units of m/s.

In summary, we have observed hysteresis at the onset of the von Kármán vortex street in a quasi-two-dimensional film. In view of prior studies of vortex shedding this result is surprising. Although the origin of the hysteresis is still not clear, we can exclude mechanical vibrations, wetting properties of the rod, and the air boundary layer as possible sources. Our preliminary experiments suggest that the effect may be connected with the fact that (a) the film is very thin in the recirculation region just before vortex shedding starts and (b) the kinematic viscosity of the film depends on its thickness [10,19,20]. The fifth-order amplitude equation is in qualitative agreement with our observations, but a simple phenomenological equation provides a better numerical fit to the experimental data in the VS regime. Theoretical explanation of our experimental findings as well as the full description of the vortex shedding continues to offer a great challenge for the future.

We thank M. Rivera for his comments and help in the experiments. We have profited from discussions and kind correspondence with M. Gharib, R. D. Henderson, D. M. Jasnow, and K. R. Sreenivasan. This material is based upon work supported by the North Atlantic Treaty Organization under Grant No. DGE-9804461 to W.I. Goldburg and V.K. Horváth. The work was also supported by NSF Grant No. DMR-9622699, NASA Grant No. 96-HEDS-01-098, and by additional support from the Hungarian Science Foundation Grant No. OTKA F17310.

[1] T. von Kármán, Gott. Nachr. Math.-Phys. Klasse **509**, 24 (1911).

[2] There has been a vast amount of research on vortex shedding behind bluff bodies; it is therefore not possible to review the literature in its entirety. For an excellent review see C. H. K. Williamson, Annu. Rev. Fluid Mech, **28**, 477 (1996).

[3] M. Provansal, C. Mathis, and L. Boyer, J. Fluid Mech. **182**, 1 (1987).

[4] K. R. Sreenivasan, P. J. Strykowski, and D. Olinger, in *ASME Forum on Unsteady Separation*, edited by K. N. Ghia (ASME Press, New York, 1987), Vol. 52, p. 1, and private communication.

[5] Y. Couder and C. Basdevant, J. Fluid Mech. **173**, 225 (1986).

[6] M. Gharib and P. Derango, Physica D **3**, 406 (1989).

[7] R. D. Henderson and D. Barkley, Phys. Fluids **8**, 1683 (1996).

[8] D. Brika and A. Laneville, J. Fluid Mech. **250**, 481 (1993).

[9] In identifying the transition time it must be taken into account that $\overline{V}(t)$ and $v_x(t)$ is measured different places. Since $\overline{V}(t)$ near the transition is ~ 50 cm/s, the related time delay (about 0.2 s) is negligible to the characteristic changing time of $\overline{V}$.

[10] M. Rivera *et al.* (unpublished).

[11] N. Berglund and H. Kunz, Phys. Rev. Lett. **78**, 1691 (1997).

[12] G. H. Goldsztein, F. Broner, and S. H. Strogatz, J. Appl. Math. **57**, 1163 (1997).

[13] S. H. Strogatz, *Nonlinear Dynamics and Chaos* (Addison Wesley, New York, 1994).

[14] M. Rivera *et al.*, Rev. Sci. Instrum. **69**, 3215 (1998).

[15] Z. Vörös, W. I. Goldburg, V. K. Horváth, J. R. Cressman, and X. L. Wu (unpublished).

[16] J. T. Stuart, J. Fluid Mech. **4**, 1 (1958).

[17] L. D. Landau, C. R. Acad. Sci. URSS **44**, 311 (1944).

[18] The mean velocity $\overline{V}$ is used here rather than the Reynolds number because of an uncertainty concerning the value of the viscosity of the flowing soap film.

[19] M. A. Rutgers *et al.*, Phys. Fluids **8**, 2847 (1996).

[20] P. Vorobieff and R. E. Ecke, Phys. Rev. E **60**, 2953 (1999).